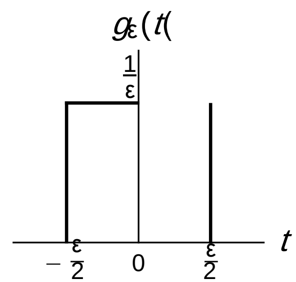
- Define

$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

- The function g_ε has a plot of the form shown below.



- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)\,dt = 1$.
- The function δ can be obtained as the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

- That is, δ can be viewed as a *limiting case of a rectangular pulse* where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

- Equivalence property. For any continuous function x and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

- Sifting property. For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0).$$

- The δ function also has the following properties:

$$\delta(t) = \delta(-t) \quad \text{and}$$

$$\delta(at) = \frac{1}{|a|}\delta(t),$$

where a is a nonzero real constant.

Representing a Rectangular Pulse Using Unit-Step

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & \text{if } a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

(i.e., $x(t)$ is a *rectangular pulse* of height one, with a *rising edge at a* and *falling edge at b*.)

- The function x can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

(i.e., the difference of two time-shifted unit-step functions.)

- Unlike the original expression for x, this latter expression for x *does not involve multiple cases*.
- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

- The idea from the previous slide can be extended to handle any function that is defined in a *piecewise manner* (i.e., via an expression involving multiple cases.(
- That is, by using unit-step functions, we can always collapse a formula involving multiple cases into a single expression.
- Often, simplifying a formula in this way can be quite beneficial.

Section 2.4

Continuous-Time (CT) Systems

- A system with input x and output y can be described by the equation

$$y = H\{x\}$$

where H denotes an operator (i.e., transformation.(

- Note that the operator H *maps a function to a function* (not a number to a number.(
- Alternatively, we can express the above relationship using the notation

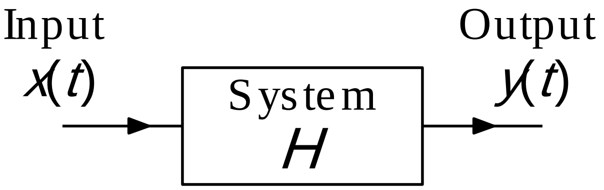
$$x \xrightarrow{H} y.$$

- If clear from the context, the operator H is often omitted, yielding the abbreviated notation

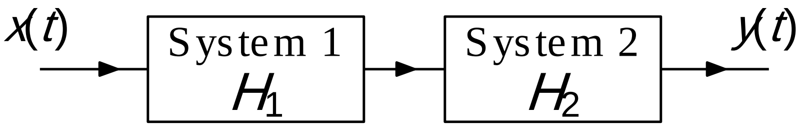
$$x \to y.$$

- Note that the symbols "$\to$" and "$=$" have *very different* meanings. The
- symbol "$\to$" should be read as "*produces*" (not as "equals.("

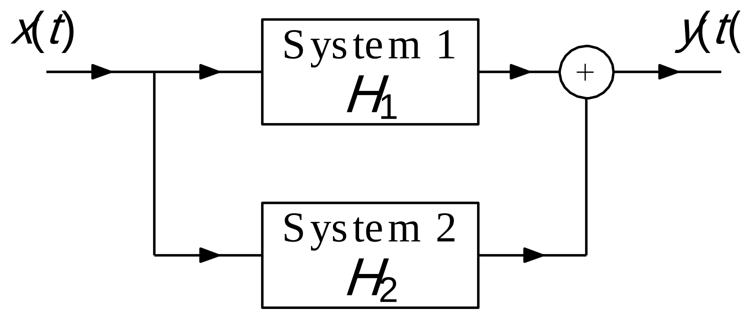
- Often, a system defined by the operator H and having the input x and output y is represented in the form of a *block diagram* as shown below.



- *Two basic ways* in which systems can be interconnected are shown below.



Series



Parallel

- A series (or cascade) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = H_2\{H_1\{x\}\}$$

- A parallel connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = H_1\{x\} + H_2\{x\}$$

Section 2.5

Properties of (CT) Systems

- A system with input x and output y is said to have memory if, for any real t_0, $y(t_0)$ depends on $x(t)$ for some $t \neq t_0$.
- A system that does not have memory is said to be memoryless.
- Although simple, a memoryless system is *not very flexible*, since its current output value cannot rely on past or future values of the input.
- A system with input x and output y is said to be causal if, for every real t_0, $y(t_0)$ does not depend on $x(t)$ for some $t > t_0$.
- If the independent variable t represents time, a system must be causal in order to be *physically realizable*.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable *need not always represent time*. For example, in some situations, the independent variable might represent position.

- The inverse of a system H is another system H^{-1} such that the combined effect of H cascaded with H^{-1} is a system where the input and output are equal.
- A system is said to be invertible if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input x can always be *uniquely* determined from its output y.
- Note that the invertibility of a system (which involves mappings between *functions*) and the invertibility of a function (which involves mappings between *numbers*) are *fundamentally different* things.
- An invertible system will always produce *distinct outputs* from any two *distinct inputs*.
- To show that a system is *invertible*, we simply find the *inverse system*. To
- show that a system is *not invertible*, we find *two distinct inputs* that result in *identical outputs*.
- In practical terms, invertible systems are "nice" in the sense that their *effects can be undone*.

- A system with input x and output y is BIBO stable if, for every bounded x, y is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|y(t)| < \infty$ for all t).
- To show that a system is *BIBO stable*, we must show that *every bounded input* leads to a *bounded output*.
- To show that a system is *not BIBO stable*, we only need to find a single *bounded input* that leads to an *unbounded output*.
- In practical terms, a BIBO stable system is *well behaved* in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time.
- Usually, a system that is not BIBO stable will have *serious safety issues*. For example, an iPod with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized Apple customer and one big lawsuit.

- A system $\mathcal{H}$ is said to be time invariant (TI) if, for every function x and every real number t_0, the following condition holds:

$$y(t-t_0) = \mathcal{H}x'(t) \quad \text{where} \quad y = \mathcal{H}x \quad \text{and} \quad x'(t) = x(t-t_0)$$

  (i.e., $\mathcal{H}$ *commutes with time shifts*).
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an *identical time shift* in the output.
- A system that is not time invariant is said to be time varying.
- In simple terms, a time invariant system is a system whose behavior *does not change* with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much *easier to design and analyze*, since their behavior does not change with respect to time.

- A system H is said to be additive if, for all functions x_1 and x_2, the following condition holds:

$$H(x_1 + x_2) = Hx_1 + Hx_2$$

)i.e., H *commutes with sums*.(

- A system H is said to be homogeneous if, for every function x and every complex constant a, the following condition holds:

$$H(ax) = aHx$$

)i.e., H *commutes with multiplication by a constant*.(

- A system that is both additive and homogeneous is said to be linear.
- In other words, a system H is *linear*, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$H(a_1x_1 + a_2x_2) = a_1Hx_1 + a_2Hx_2$$

)i.e., H *commutes with linear combinations*.(

- The linearity property is also referred to as the superposition property.
- Practically speaking, linear systems are much *easier to design and analyze* than nonlinear systems.

Part 3

Continuous-Time Linear Time-Invariant (LTI) Systems